ДЗ 5 4/6

№5 (тема 6)

$u_2 = x^2 - y^2 + xy$

$\Delta u_2 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$ – гармоническая

$u_4 = \cos x \cdot \operatorname{sh} y - \operatorname{sh} x \cdot \sin y$

$\Delta u_4 = (-\cos x \cdot \operatorname{sh} y - \operatorname{sh} x \sin y) + (\cos x \operatorname{sh} y + \operatorname{sh} x \cdot \sin y) = 0$ – гармоническая

$u_8 = e^{x^2+y^2}$

$\frac{\partial u}{\partial x} = 2x \cdot e^{x^2+y^2}$ $\quad \frac{\partial^2 u}{\partial x^2} = 2 \cdot e^{x^2+y^2} + 2x \cdot 2x \cdot e^{x^2+y^2} = (2+4x^2) e^{x^2+y^2}$

$\Delta u_8 = (2+4x^2) e^{x^2+y^2} + (2+4y^2) e^{x^2+y^2} = (4+4x^2+4y^2) e^{x^2+y^2} > 0$

– не гармоническая

№3 (тема 7)

$|u(r,\varphi)| < const$ $\quad r > a$ $\quad 0 \le \varphi < 2\pi$

$\Delta u = 0$

$u(a,\varphi) = f(\varphi)$ $\quad 0 \le \varphi < 2\pi$ $\quad f(0) = f(2\pi)$ $\quad f \in C[0, 2\pi]$

$u(r,\varphi) = R(r) \cdot \Phi(\varphi)$

$\Delta u = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \varphi^2} = 0$

$\Phi \frac{1}{r} \frac{\partial}{\partial r}(r R'(r)) + \frac{1}{r^2} \Phi'' R = 0$

$$\frac{r \cdot \frac{\partial}{\partial r}\left(r \frac{\partial R}{\partial r}\right)}{R} = \frac{\Phi''}{\Phi} = \lambda$$

1) $\begin{cases} \Phi'' + \lambda \Phi = 0 \\ \Phi(\varphi) = \Phi(\varphi + 2\pi) \end{cases}$

$\Phi(\varphi) = A \cos(\sqrt{\lambda}\varphi) + B \sin(\sqrt{\lambda}\varphi)$

$\sqrt{\lambda} = n = 0, 1, 2 \ldots$

2) $r(R' + r \cdot R'') = \lambda R$

$R'' r^2 + r R' - \lambda R = 0$

$R = \ln r \quad \lambda = 0$

$R = A_0 = const \ne 0$

$R = r^{\sqrt{\lambda}} + r^{-\sqrt{\lambda}}, \quad \lambda > 0$

т.к. $|u| < const$, то

$$u = \sum_{n=0}^{\infty} r^{-n} (A_n \cos n\varphi + B_n \sin n\varphi)$$

$u(a,\varphi) = \sum_{n=0}^{\infty} a^{-n} (A_n \cos n\varphi + B_n \sin n\varphi)$

$A_n = a^n \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$ $\quad B_n = a^n \cdot \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$

$A_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$ $\quad B_0 = 0.$

№ 5

$$\begin{cases} \Delta u = 0 & r > a \\ \left.\frac{\partial u}{\partial r}\right|_{r=a} = f(\varphi) & 0 \le \varphi < 2\pi \end{cases}$$

Аналогично № 3

$$u(r,\varphi) = \sum_{n=0}^{\infty} r^{-n}(A_n \cos n\varphi + B_n \sin n\varphi)$$

$$\frac{\partial u}{\partial r} = \sum_{n=0}^{\infty} (-n) r^{-n-1}(A_n \cos n\varphi + B_n \sin n\varphi)$$

$$\frac{\partial u}{\partial r}(a,\varphi) = \sum_{n=0}^{\infty} (-n) a^{-n-1}(A_n \cos n\varphi + B_n \sin n\varphi) = f(\varphi)$$

$$A_n = -\frac{1}{n} a^{n+1} \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\cos n\varphi \, d\varphi \qquad n > 1$$

$$B_n = -\frac{1}{n} a^{n+1} \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\sin n\varphi \, d\varphi$$

A_0 - любое $\qquad B_0$ - не участвует, т.к $\neq 0$

$$u(r,\varphi) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\varphi + B_n \sin n\varphi) r^{-n}$$

№ 6

Из № 5 $A_n = -n \cdot a^{n+1} \frac{1}{\pi}\int_0^{2\pi} 1 \cos n\varphi \, d\varphi = 0 = B_n$

$\Rightarrow u(r,\varphi) = A_0$ - не подходит.

! не выполнено $\int_0^{2\pi} \left.\frac{\partial u}{\partial r}\right|_{r=a} d\varphi = 2\pi \neq 0$

№ 7

а) $u|_{r=1} = \sin 3\varphi$, $\Delta u = 0$ $\quad 0 \le r < 1$, $0 \le \varphi < 2\pi$

$$u(r,\varphi) = \sum_{n=0}^{\infty} r^n (A_n \sin n\varphi + B_n \cos n\varphi)$$

$$u(1,\varphi) = \sum_{n=0}^{\infty} (A_n \sin n\varphi + B_n \cos n\varphi) = \sin 3\varphi$$

$A_3 = 1 \quad A_i = 0 \quad i \neq 3 \qquad B_i = 0 \qquad i \in \mathbb{N} \cup \{0\}$

$$u(r,\varphi) = r^3 \sin 3\varphi$$

б) $u|_{r=1} = \cos^2\varphi$, $\Delta u = 0$ $\quad 0 \le r < 1 \quad 0 \le \varphi < 2\pi$

$$\cos^2\varphi = \frac{1 - \cos 2\varphi}{2}$$

$\Rightarrow B_0 = \frac{1}{2} \quad B_2 = -\frac{1}{2} \quad A_i, B_i = 0$

$$u(r,\varphi) = \frac{1}{2} + -\frac{1}{2} r^2 \cos 2\varphi$$

№ 8

б) $\left.\frac{\partial u}{\partial r}\right|_{r=1} = \sin\varphi + \cos\varphi$

$\int_0^{2\pi} (\sin\varphi + \cos\varphi)\, d\varphi = 0$ ✓

$\Delta u = 0 \qquad 0 \le r \le 1 \qquad 0 \le \varphi < 2\pi$

$u(r,\varphi) = \sum_{n=0}^{\infty} r^{-n}(A_n\cos n\varphi + B_n \sin n\varphi)$

$\left.\frac{\partial u}{\partial r}\right|_{r=1} = \sum_{n=1}^{\infty} (-n)(A_n\cos n\varphi + B_n\sin n\varphi) = \cos\varphi + \sin\varphi$

$\Rightarrow A_1 = -1 \qquad B_1 = -1 \qquad B_i = A_i = 0 \quad i > 1$

$u(r,\varphi) = A_0 + \left(-\frac{1}{r}\right)(\cos\varphi + \sin\varphi)$

в) $\left.\frac{\partial u}{\partial r}\right|_{r=1} = \cos^2\varphi$

$\int_0^{2\pi} \cos^2\varphi\, d\varphi \ne 0 \Rightarrow$ решения нет

№ 10 $\left.\frac{\partial u}{\partial(-r)}\right|_{r=1} = \cos\varphi$

$\frac{\partial u}{\partial(-r)} = -\frac{\partial u}{\partial r}$

$\Rightarrow A_1 = 1, \quad A_i = B_i = 0$ — остальные

$u(r,\varphi) = A_0 + \left(\frac{1}{r}\right)\cos\varphi$

№ 13
$$\begin{cases} \Delta u = 0 & 0 < r < a \\ & 0 < \varphi < \alpha < 2\pi \\ u(r,0) = 0 & \\ u(r,\alpha) = 0 & 0 \le r \le \alpha \\ u(a,\varphi) = f(\varphi), & 0 < \varphi < \alpha \end{cases}$$

$u(r,\varphi) = R(r)\,\Phi(\varphi)$

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(0) = 0 \\ \Phi(\alpha) = 0 \end{cases}$$

$\Phi_n = A_n \cos\sqrt{\lambda_n}\varphi + B_n \sin\sqrt{\lambda_n}\varphi$

$\Phi_n(0) = 0 \Rightarrow A_n = 0$

$\Phi_n(\alpha) = 0 \Rightarrow \sqrt{\lambda_n}\cdot\alpha = \pi n \qquad \lambda_n = \left(\frac{\pi n}{\alpha}\right)^2$

$\lambda_n > 0$

$R(\varphi) = A_n r^{-\sqrt{\lambda_n}} + B_n r^{+\sqrt{\lambda_n}}$

т.к. R непр в $0 < r < a$, то $A_n = 0$.

$u(r,\varphi) = \sum_{n=1}^{\infty} A_n r^{+\frac{\pi n}{\alpha}} \sin\frac{\pi n\varphi}{\alpha}$

$u(a,\varphi) = f(\varphi) = \sum_{n=1}^{\infty} A_n a^{+\frac{\pi n}{\alpha}} \sin\frac{\pi n\varphi}{\alpha}$

$A_n = a^{-\frac{\pi n}{\alpha}} \cdot \frac{2}{\alpha}\int_0^{\alpha} f(\varphi)\sin\frac{\pi n\varphi}{\alpha}\, d\varphi$

№14 Д35 4/6

$$\begin{cases} \Delta u = 0 & 0 < r < 2 \quad 0 < \varphi < 1 \\ u(r,0) = 0 \quad u(r,1) = 0 & 0 \le r \le 2 \\ u(2,\varphi) = \sin(3\pi\varphi) & 0 < \varphi < 1 \end{cases}$$

$$u(r,\varphi) = \sum_{n=1}^{\infty} r^n A_n \sin\left(\frac{\pi n \varphi}{1}\right)$$

$$u(2,\varphi) = \sum_{n=1}^{\infty} 2^n A_n \sin(\pi n \varphi) = \sin(3\pi\varphi)$$

$$\Rightarrow A_3 = \frac{1}{2^3} \qquad A_i = 0$$

$$u(r,\varphi) = \frac{r^3}{2^3} \sin(3\pi\varphi)$$

№17

$$\Delta u = \sin\varphi \qquad 0 \le r < a \qquad 0 \le \varphi < 2\pi$$

$$u(a,\varphi) = a^2 \cdot \sin 2\varphi, \qquad 0 \le \varphi < 2\pi$$

$$\tilde{u} = r^2 \sin\varphi$$

$$\frac{\partial\tilde{u}}{\partial r} = 2r\sin\varphi, \quad r\frac{\partial\tilde{u}}{\partial r} = 2r^2\sin\varphi \qquad \frac{\partial}{\partial r}\left(r\frac{\partial\tilde{u}}{\partial r}\right) = 4r\sin\varphi \qquad \frac{1}{r}\left(\frac{\partial}{\partial r} r\frac{\partial\tilde{u}}{\partial r}\right) = 4\sin\varphi$$

$$\frac{1}{r^2}\frac{\partial^2\tilde{u}}{\partial\varphi^2} = -\frac{r^2\sin\varphi}{r^2} = -\sin\varphi$$

$$\Rightarrow \quad \Delta\tilde{u} = 4\sin\varphi - \sin\varphi = 3\sin\varphi$$

$\Rightarrow \quad U = \frac{1}{3} r^2 \sin\varphi$ — частное решение $\Delta u = \sin\varphi$

$$u = v + U$$

$$\Delta v = 0$$

$$v(a,\varphi) + \frac{1}{3} a^2 \sin\varphi = a^2 \sin 2\varphi$$

$$\begin{cases} v(a,\varphi) = a^2 \sin\varphi\left(2\cos\varphi - \frac{1}{3}\right) = a^2 \sin 2\varphi - \frac{1}{3} a^2 \sin\varphi \\ \Delta v = 0 \end{cases}$$

$$v(r,\varphi) = \sum_{n=0}^{\infty} r^n (A_n \sin n\varphi + B_n \cos n\varphi)$$

$$r(a,\varphi) = \sum_{n=0}^{\infty} a^n (A_n \sin n\varphi + B_n \cos n\varphi) = a^2 \sin 2\varphi - \frac{1}{3} a^2 \sin\varphi$$

$$A_1 \cdot a^1 = -\frac{1}{3} a^2 \quad \Rightarrow \quad A_1 = -\frac{1}{3} a$$

$$A_2 \cdot a^2 = a^2 \Rightarrow A_2 = 1 \qquad A_i = B_i = 0$$

$$u = \frac{r^2}{3} \sin\varphi - \frac{1}{3} a \sin\varphi \cdot r + r^2 \sin 2\varphi$$

Д3 5 5/6

№ 18

$\Delta u = 1 \qquad 0 < a < r < b \qquad 0 \le \varphi < 2\pi$

$u(a,\varphi) = a^2 \sin\varphi \qquad 0 \le \varphi < 2\pi$

$u(b,\varphi) = b^2 \cos\varphi$

$U = Ar^2$

$\frac{\partial U}{\partial r} = 2Ar \qquad r\frac{\partial U}{\partial r} = 2r^2 A \qquad \dots \qquad \frac{1}{r}\left(\frac{\partial}{\partial r} r \frac{\partial U}{\partial r}\right) = 4 \cdot A$

$\Rightarrow A = \frac{1}{4} \qquad U = \frac{r^2}{4}$ - частное решение

$u = v + U$

$$\begin{cases} \Delta v = 0 \\ v(a,\varphi) + \frac{a^2}{4} = a^2 \sin\varphi \\ v(b,\varphi) + \frac{b^2}{4} = b^2 \cos\varphi \end{cases}$$

$v(r,\varphi) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} (A_n r^{-n} + B_n r^n)(C_n \sin n\varphi + D_n \cos n\varphi)$

$v(a,\varphi) = A_0 + B_0 \cdot \ln a + \sum_{n=1}^{\infty} (A_n a^{-n} + B_n a^n)(C_n \sin n\varphi + D_n \cos n\varphi) = a^2 \sin\varphi + \frac{a^2}{4}$

$A_0 + B_0 \ln a = \frac{a^2}{4} \qquad \left(\frac{A_1}{a} + B_1 \cdot a\right) \cdot C_1 = a^2 \qquad \left(\frac{A_1}{a} + B_1 a\right) D_1 = 0$

$n > 1 \quad (A_n a^{-n} + B_n a^n) C_n = 0$

$n > 6 \quad (A_n a^{-n} + B_n a^n) D_n = 0$

$v(b,\varphi) = A_0 + B_0 \ln b + \sum_{n=1}^{\infty} (A_n b^{-n} + B_n b^n)(C_n \sin n\varphi + D_n \cos n\varphi) = b^2 \cos\varphi + \frac{b^2}{4}$

$A_0 + B_0 \ln b = \frac{b^2}{4}$

$\left(\frac{A_1}{b} + B_1 b\right) D_1 = b^2 \qquad \left(\frac{A_1}{b} + B_1 b\right) C_1 = 0$

$\left(\frac{A_n}{b^n} + B_n b^n\right) C_n = 0 \qquad \left(\frac{A_n}{b^n} + B_n b^n\right) D_n = 0$

$n > 0 \qquad n > 1$

$\Rightarrow C_n = D_n = 0 \qquad n > 1$

$B_0 = \frac{b^2 - a^2}{4\ln(\frac{b}{a})} \qquad A_0 = \frac{b^2 \ln a - a^2 \ln b}{4 \cdot \ln(a \cdot b)}$

$A_1 C_1 = A \qquad B_1 C_1 = B \qquad A_1 D_1 = C \qquad B_1 D_1 = D$

$\frac{A}{a} + Ba = a^2 \qquad \frac{C}{a} + Da = 0 \qquad \frac{A}{b} + Bb = 0 \qquad \frac{C}{b} + bD = b^2$

$$\left[\begin{array}{cccc|c} \frac{1}{a} & a & 0 & 0 & a^2 \\ 0 & 0 & \frac{1}{a} & a & 0 \\ \frac{1}{b} & b & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{b} & b & b^2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & a^2 & 0 & 0 & a^3 \\ 0 & 0 & 1 & a^2 & 0 \\ 1 & b^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & b^2 & b^3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & a^2 & 0 & 0 & a^3 \\ 0 & 0 & 1 & a^2 & 0 \\ 0 & b^2 - a^2 & 0 & 0 & -a^3 \\ 0 & 0 & 0 & b^2 - a^2 & +b^3 \end{array}\right]$$

$\Rightarrow B = \frac{-a^3}{b^2 - a^2} \qquad A = a^3 + \frac{a^5}{b^2 - a^2} = \frac{a^3 b^2 - a^5 + a^5}{b^2 - a^2} = \frac{a^3 b^2}{b^2 - a^2}$

$D = \frac{b^3}{b^2 - a^2} \qquad C = \frac{-a^2 \cdot b^3}{b^2 - a^2} \Rightarrow u(r,\varphi) = \frac{r^2}{4} + \frac{b^2 - a^2}{4\ln(\frac{b}{a})} \ln r + \frac{b^2 \ln a - a^2 \ln b}{4 \ln(ab)} + \frac{1}{r}\left(\frac{a^3 b^2}{b^2 - a^2}\right)\sin\varphi + r\left(\frac{-a^3}{b^2 - a^2}\right)\sin\varphi + \frac{1}{r}\left(\frac{-a^2 b^3}{b^2 - a^2}\right)\cos\varphi + r\left(\frac{b^3}{b^2 - a^2}\right)\cos\varphi$ ✗

№ 19

$$\begin{cases} \Delta u = 0 & 0 < x < \infty \quad 0 < y < a \\ u(0, y) = f(y) & 0 \le y \le a \\ u(x, 0) = f(0) = const & 0 \le x \le \infty \\ u(x, a) = f(a) = const & \\ |u| < const & \end{cases}$$

$$u = v + \underbrace{\frac{f(a) - f(0)}{a} y + f(0)}_{u}$$

$$\Rightarrow \begin{cases} \Delta v = 0 \\ v(0, y) = f(y) - u(y) = g(y) \\ v(x, 0) = 0 \\ v(x, a) = 0 \end{cases}$$

$$v = X(x) \cdot Y(y)$$

$X'' - \lambda X = 0$ $\qquad$ $Y'' + \lambda Y = 0$

$|X| < const$ $\qquad$ $Y(0) = 0 \quad Y(a) = 0$

$X = C_n e^{\sqrt{\lambda} x} + D_n e^{-\sqrt{\lambda} x}$ $\qquad$ $Y_n = A_n \sin\sqrt{\lambda_n} y + B_n \cos\sqrt{\lambda_n} y$

$Y(0) = 0 \Rightarrow B_n = 0$

$C_n = 0$, т.к. $|X| < const$ $\qquad$ $Y(a) = 0 \Rightarrow \sqrt{\lambda_n} a = \pi n$ $\qquad$ $\lambda_n = \left(\frac{\pi n}{a}\right)^2$

$$\Rightarrow v = \sum_{n=1}^{\infty} e^{-\frac{\pi n}{a} x} \left(A_n \sin\frac{\pi n y}{a} + B_n \cos\frac{\pi n y}{a}\right)$$

$$A_n = \frac{2}{a} \int_0^a g(y) \cdot \sin\frac{\pi n y}{a} dy \qquad B_n = \frac{2}{a} \int_0^a g(y) \cos\frac{\pi n y}{a} dy$$

№ 26

$$\begin{cases} \Delta u = \sin 2x & 0 < x < \frac{\pi}{2} \quad 0 < y < \infty \\ u(0, y) = 0 & 0 \le y \le \infty \\ u(\frac{\pi}{2}, y) = 0 & 0 \le y \le \infty \\ u(x, 0) = \sin 4x & 0 \le x \le \frac{\pi}{2} \end{cases}$$

Замена $u = v - \frac{1}{4} \sin 2x$

$$\Rightarrow \begin{cases} \Delta v = 0 \\ v(0, y) = 0 \\ v(\frac{\pi}{2}, y) = 0 \\ v(x, 0) = \sin 4x + \frac{1}{4} \sin 2x \end{cases}$$

Из 19 $x \leftrightarrow y$ $\quad v = \sum_{n=1}^{\infty} e^{-\frac{\pi n y}{\pi/2}} \left(A_n \sin\frac{\pi n x}{\pi/2} + B_n \cos\frac{\pi n x}{\pi/2}\right)$

$$v(x, 0) = \sin 4x + \frac{1}{4} \sin 2x = \sum_{n=1}^{\infty} A_n \sin(2nx) + B_n \cos(2nx)$$

$\Rightarrow A_1 = \frac{1}{4} \quad A_2 = 1 \quad B_i = A_j = 0. \quad i > 0 \quad j > 2$

$$u(x, y) = -\frac{1}{4} \sin 2x + e^{-2y} \frac{1}{4} \sin(2x) + e^{-4y} \sin(4x)$$